Weak Convergence of Martingales and Its Application to Nonlinear Cointegrating Regression Model

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Outline

The Semiparametric Nonlinear Cointegrating Regression Model

$$y_t = f(x_t) + u_t,$$

where

- x_t : Non-stationary regressor,
- $f(\cdot)$: An unknown function on $\mathbb{R}$,
- u_t : An equilibrium error satisfying $\mathbb{E}[u_t|x_t] = 0$.

The Model (cont.)

- $\mathcal{M} = \{g(x,\theta) : \theta \in \Theta_0\}$: A family of real functions indexed by a vector $\theta = (\theta_1, \ldots, \theta_m)'$ of unknown parameters lying in the compact parameter space $\Theta_0 \subset \mathbb{R}^m$.
- The problem: Testing the hypothesis $f(\cdot) \in \mathcal{M}$, i.e.,

$$H_0 : \mathbb{E}[y_t - g(x_t, \theta_0) | x_t] = 0, \quad \text{for some } \theta_0 \in \Theta_0.$$

A Remark

In the literature, the null hypothesis H_0 is close to the problem of testing the martingale difference, cf. Stute (1997), Deo (2000), Escanciano (2006) for stationary cases and Park and Whang (2005), Escanciano (2007) and Phillips and Jin (2014) for non-stationary cases.

Test Statistic for H_0

We introduce the test statistic for H_0 in two steps.
Step 1: Define the marked empirical process $\alpha_n(x)$ by

$$\alpha_n(x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left[y_k - g(x_k, \widehat{\theta}_n) \right] \mathbb{1}(x_k/d_n \leq x),$$

where

- $0 < d_n \to \infty$: A sequence of constants which will be specified later,
- $\widehat{\theta}_n$: Nonlinear least square estimator of θ_0 defined by

$$\widehat{\theta}_n := \mathrm{argmin}_{\theta \in \Theta_0} \sum_{k=1}^{n} (y_k - g(x_k, \theta))^2.$$

Test Statistic for H_0 (cont.)

Step 2: As in Stute (1997), the test statistic for H_0 is

$$S_n = \sup_{x \in \mathbb{R}} |\alpha_n(x)| = \sup_{x \in \mathbb{R}} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left[y_k - g(x_k, \widehat{\theta}_n) \right] \mathbb{1}(x_k \leq x) \right|.$$

Goal: We want to obtain the asymptotic distribution of S_n under H_0.

Assumptions on x_t and u_t

Let ε_j, $j \in \mathbb{Z}$ be a sequence of i.i.d. r.v.'s with $\mathbb{E}[\varepsilon_0] = 0$, $\mathbb{E}[\varepsilon_0^2] = 1$ and $\lim_{|t|\to\infty} |t|^\eta \left|\mathbb{E}[e^{it\varepsilon_0}]\right| < \infty$ for some $\eta > 0$, and let ξ_j, $j \geq 1$ be a linear process defined by

$$\xi_j = \sum_{k=0}^{\infty} \phi_k \varepsilon_{j-k},$$

Assumptions on x_t and u_t (cont.)

where the coefficients $\phi_k,\ k \geq 0$ satisfy one of the following conditions

- **LM.** $\phi_k \sim k^{-\mu} L(k)$, where $1/2 < \mu < 1$ and $L(k)$ is a slowly varying function at ∞
- **SM.** $\sum_{k=0}^{\infty} |\phi_k| < \infty$ and $\phi := \sum_{k=0}^{\infty} |\phi_k| \neq 0$.

Fact: Define $d_n^2 = \mathrm{Var}\left(\sum_{j=1}^{n} \xi_j\right)$, by Wang et al. (2003), we have

$$d_n^2 \sim c_\mu n^{3-2\mu} L(n), \text{under } \mathbf{LM}, \ d_n^2 \sim \phi^2 n, \text{under } \mathbf{SM}.$$

Assumptions on x_t and u_t (cont.)

Assumption 2.1. $x_k = \gamma x_{k-1} + \xi_k$, where $x_0 = 0$ and $\gamma = 1 - \tau/n$ for some $\tau \geq 0$. (x_t is a near integrated short/long memory linear process.)

Assumption 2.2. (i) For each $k \geq 1$, $\mathbb{E}[u_k|\mathcal{F}_{k-1}] = 0$ and $\sup_{k\geq 1} \mathbb{E}[|u_k|^{2+\eta}|\mathcal{F}_{k-1}] < \infty$ for some $\eta > 0$, where $\mathcal{F}_k$ is an increasing sequence of σ-fields such that $x_k \in \mathcal{F}_k$; (ii) we have

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \varepsilon_k, \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \varepsilon_{-k}, \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} u_k \right) \Rightarrow (B_{1t}, B_{2t}, B_{3t}) \text{, on } D_{\mathbb{R}^3}[0, \infty),$$

where $(B_{1t}, B_{2t}, B_{3t})_{t\geq 0}$ is a three dimensional Brownian motion with covariance matrix Ω.

Assumptions on x_t and u_t (cont.)

Remark: Assumption 2.2(ii) ensures the following joint convergence [cf. Buchmann and Chan (2007)]:

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} u_k, \frac{x_{[nt]}}{d_n} \right) \Rightarrow (B_{3t}, X_t), \text{ on } D_{\mathbb{R}^2}[0, 1],$$

where $X_t \in \mathcal{F}_t$, and

$$X_t = \int_0^t e^{-\tau(t-s)} dW(s) = W(t) + \tau \int_0^t e^{-\tau(t-s)} W(s) ds,$$

Assumptions on x_t and u_t (cont.)

where $W(t) = B_{1t}$ under **SM**; and $W(t) = W_H(t)$, where

$$W_H(t) = c_H \left(\int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dB_{2s} + \int_0^t (t-s)^{H-\frac{1}{2}} dB_{1s} \right),$$

which is a fractional Brownian motion with Hurst index $H = 3/2 - \mu \in (1/2, 1)$, under **LM**.

Asymptotics of S_n : Integrable Regression Function

Assumption 2.3. There exist a bounded function $h(x)$ satisfying $h(x) \downarrow h(0) = 0$, as $x \downarrow 0$, and a bounded integrable function $T(x)$ such that for all $\theta,\ \theta_0 \in \Theta_0$,

$$|g(x,\theta) - g(x,\theta_0)| \leq h(\|\theta - \theta_0\|)T(x),$$

and

$$\int_{-\infty}^{\infty} (g(s,\theta) - g(s,\theta_0))^2\, ds > 0 \text{ for all } \theta \neq \theta_0.$$

Some Examples:

$$\theta_1 |x|^{\theta_2} \mathbb{1}(x \in [a,\ b]),\ e^{-\theta|x|},\ e^{\theta|x|}/(1 + e^{\theta|x|}).$$

Asymptotic Distribution of S_n under H_0: Intrgrable Regression Function (Wang, W., Zhu, 2018)

Theorem

Suppose that Assumptions 2.1-2.3 hold. Then under H_0

$$S_n \to_D \sup_{x \in \mathbb{R}} |\alpha(x)|,$$

where $\alpha(x) = \int_0^1 \mathbb{1}(X_t \leq x) dB_{3t}$.

A Remark

Remark: From the above theorem, we can derive that, under H_0

$$S_n = \sup_{x\in\mathbb{R}} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} u_k \mathbb{1}(x_k \leq x) \right| + o_P(1),$$

indicating that there is no estimation effect under H_0 when the regressor x_t is nonstationary and the regression function $g(x,\theta)$ is integrable. This is due to the fact that nonstationarity weakens the signal when the transformation function is integrable, which differs from the stationary situation, see, e.g., Stute (1997), Escanciano (2006, 2007), Ling and Tong (2011).

Asymptotics of S_n : Non-Integrable Regression Function

Assumption 2.3 is somehow restrictive, which exclude some practically useful models such as $g(x, \theta) = \theta x$ and $(x - \theta)^2$. The following Assumptions 2.5-2.6 remove the restriction on the boundedness and integrability of $T(x)$, but impose more smooth conditions on $g(x, \theta)$.

Let

$$\dot{g}_i = \frac{\partial g(x, \theta)}{\partial \theta_i}, \ \ddot{g}_{ij} = \frac{\partial^2 g(x, \theta)}{\partial \theta_i \theta_j}, \ 1 \leq i \leq j \leq m.$$

Asymptotics of S_n : Non-Integrable Regression Function (cont.)

Assumption 2.5. Let $p(x, \theta)$ be one of g, $\dot{g}_i$, $\ddot{g}_{ij}$. There exists a real function $T_p : \mathbb{R} \to \mathbb{R}$ such that

- $|p(x, \theta) - p(x, \theta_0)| \leq A_p(\|\theta - \theta_0\|)T_p(x)$, where $A_p(x)$ is a real function satisfying $A_p(t) \downarrow A_p(0) = 0$ as $t \downarrow 0$;
- for any bounded x

$$\sup_{\theta \in \Theta_0} \frac{|p(\lambda x, \theta) - v_p(\lambda)h_p(x, \theta)|}{T_p(\lambda x)} = o(1),$$

as $\lambda \to \infty$, where for each $\theta \in \Theta_0$, $h_p(x, \theta)$ is a locally bounded function, and $v_p(\lambda)$ is a positive real function bounded away from 0 as $\lambda \to \infty$.

Asymptotics of S_n : Non-Integrable Regression Function (cont.)

- $T_p(\lambda x) \leq C v_p(\lambda)\,(1 + |x|^\gamma)$ as $|\lambda x| \to \infty$ for some $\gamma > 0$.

Assumption 2.6. We have

$$\sup_{1 \leq i,j \leq m} \frac{v(n)\ddot{v}_{ij}(n)}{\dot{v}_i(n)\dot{v}_j(n)} < \infty, \quad \int_{|s|<\delta} \dot{h}(s,\theta_0)\dot{h}(s,\theta_0)' ds > 0 \text{ for some } \delta > 0,$$

where

$$v(n) = v_g(d_n), \;\; \dot{v}_i(n) = v_{\dot{g}_i}(d_n), \;\; \ddot{v}_{ij}(n) = v_{\ddot{g}_{ij}}(d_n),$$

$$\dot{h}(s,\theta_0) = \left(h_{\dot{g}_1}(s,\theta_0), \ldots, h_{\dot{g}_m}(s,\theta_0)\right)'.$$

Asymptotics of S_n : Non-Integrable Regression Function (Wang, W., Zhu, 2018)

Theorem

Suppose that Assumptions 2.1-2.2 and 2.5-2.6 hold. Then, under H_0,

$$S_n \to_D \sup_{x\in\mathbb{R}} |\beta(x)| ,$$

where

$$\beta(x) = \int_0^1 \mathbb{1}(X_t \le x) dB_{3t} - \int_0^1 \mathbb{1}(X_s \le x)\Psi_s' ds \left(\int_0^1 \Psi_u \Psi_u' du\right)^{-1} \int_0^1 \Psi_t dB_{3t},$$

with $\Psi_t = \dot{h}(X_t, \theta_0)$.

Remark

Some typical functions that satisfy Assumptions 2.5-2.6:

$$g(x,\theta) = (x+\theta)^2,\ \frac{\theta e^x}{1+e^x},\ \theta \log|x|,\ \theta|x|^\alpha\ (\alpha \text{ is fixed.})$$

Weak Convergence of Martingales

In this section, we consider weak convergence for a class of martingales, which provides a technique tool in establishing the asymptotic distributions of S_n under H_0 (Theorem 2.1 and Theorem 2.2).

Weak Convergence of Martingales (cont.)

Let $\{u_k, y_{nk}\}_{k\geq 1, n\geq 1}$ be a triangular array on $\mathbb{R}\times\mathbb{R}$. Write

$$M_n(-\infty) = 0,\ \ M_n(x) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} u_k \mathbb{1}(y_{nk}\leq x)\ \ -\infty < x \leq \infty,$$

$$M_{n1}(-\infty) = 0,\ \ M_{n1}(x) = \frac{1}{n}\sum_{k=1}^{n} g_1(y_{nk})\mathbb{1}(y_{nk}\leq x)\ -\infty < x \leq \infty,$$

$$M_{n2} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} u_k g_2(y_{nk}),\ \ M_{n3} = \frac{1}{n}\sum_{k=1}^{n} g_3(y_{nk}),$$

where $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$ are locally bounded functions on $\mathbb{R}$.

Assumptions

Assumption 3.1. $\{u_k, \mathcal{F}_k\}_{k\geq 1}$ forms a martingale difference satisfying $\sup_{k\geq 1} \mathbb{E}[|u_k|^{2+\eta}|\mathcal{F}_{k-1}] < \infty$ for some $\eta > 0$.
Assumption 3.2. y_{nk} is adapted to $\mathcal{F}_{k-1}$ for each $n \geq 1$, and

$$\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} u_j, y_{n,[nt]} \right) \Rightarrow (U_t, Y_t), \quad \text{on } D_{\mathbb{R}^2}[0, 1].$$

Assumptions (cont.)

Assumption 3.3. There exists a $d \in (0, 1)$ such that (i) for each $k \geq 1$ and $n \geq 1$, $(n/k)^d y_{nk}$ has a density $h_{nk}(x)$ which is uniformly bounded by a constant K; (ii) for $j \geq 1$, $n \geq 1$ and $k \geq j + n_0$ for some positive integer n_0, conditioning on $\mathcal{F}_{nj} = \sigma(y_{n1}, \ldots, y_{nj})$, $[n/(k-j)]^d (y_{nk} - y_{nj})$ has a density $h_{nkj}(x)$ which is uniformly bounded by a constant K and

$$\sup_{u \in \mathbb{R}} \left| h_{nkj}(u+t) - h_{nkj}(u) \right| \leq C \min\{|t|, 1\},$$

for $t \in \mathbb{R}$, where $C > 0$ is a constant.

Comments on Assumptions

Assumptions 3.1-3.2 are weak and close to be necessary. Assumption 3.3 is a special version of the strong smooth condition introduced in Wang (2015), which is used to offset the effect of the indicator function $\mathbb{1}(y_{nk} \leq x)$.

Weak Convergence Theorem of Martingales (Wang, W., Zhu, 2018)

Theorem

Suppose that Assumptions 3.1-3.3 hold. Then

$$(M_n(x), M_{n1}(x), M_{n2}, M_{n3}) \Rightarrow (M(x), M_1(x), M_2, M_3) \text{ on } D_{\mathbb{R}^4}[-\infty, \infty],$$

where

$$M(x) = \int_0^1 \mathbb{1}(Y_t \leq x) dU_t, \quad M_1(x) = \int_0^1 g_1(Y_t)\mathbb{1}(Y_t \leq x) dt,$$

$$M_2 = \int_0^1 g_2(Y_t) dU_t, \quad M_3 = \int_0^1 g_3(Y_t) dt.$$

Proof

The proof for the convergence of finite dimensional distribution is followed from some standard methods (Cramer-Wold device). The proof for the tightness is difficult, and our method is different from those in the literature, e.g., Park and Whang (2005), Escanciano (2007) and Phillips and Jin (2014).

Proof for the Tightness of $\{M_{n1}(x)\}_{n\geq 1}$ on $D_{\mathbb{R}}[-\infty, \infty]$ (Sketch)

For $s = 1, 2, \ldots,$ define

$$M_{n1}^{(s)}(\theta) = \frac{1}{n}\sum_{k=1}^{n} g_1(y_{nk})\mathbb{1}(y_{nk} \leq \theta_s),$$

where $\theta_s = j2^{-s}$ if $\theta \in [j2^{-s}, (j+1)2^{-s}]$, $j \in \mathbb{Z}$. Then, it suffices to prove that, for every $\varepsilon, \varepsilon_1 > 0$, there exists a positive integer k_0 such that

Proof for the Tightness (cont.)

$$\limsup_{n\to\infty} \mathbb{P}\left[\sup_{\theta\in[-1,1]} |M_{n1}(\theta) - M_{n1}^{(k_0)}(\theta)| \geq \varepsilon\right] \leq \varepsilon_1, \qquad (*)$$

$$\limsup_{n\to\infty} \mathbb{P}\left[\sup_{\theta\in[-1,1]} |M_{n1}^{(k_0+1)}(\theta) - M_{n1}^{(k_0)}(\theta)| \geq \varepsilon\right] \leq \varepsilon_1. \qquad (**)$$

Proof for the Tightness (cont.)

To prove $(*)$, notice that $g_1(t)$ is locally bounded, we have that, for any $\theta \in [-1, 1]$,

$$|M_{n1}(\theta) - M_{n1}^{(k_0)}(\theta)| \leq \frac{C}{n} \max_{-2^{k_0} \leq j \leq 2^{k_0}} W_n(j, k_0),$$

where

$$W_n(j, k_0) = \sum_{k=1}^{n} \mathbb{1}(j2^{-k_0} \leq y_{nk} \leq (j+1)2^{-k_0}).$$

Proof for the Tightness (cont.)

Lemma

Suppose that Assumption 3.3 holds, then for any j and s,

$$\mathbb{E}\left[(W_n(j,s))^m\right] \leq C_0^m m!(1 + n2^{-s}),$$

where $C_0 > 0$ is a constant independent of n, j and m.

[cf. Lemma 2.5 of Wang (2015)]

Proof for the Tightness (cont.)

Now, as $n \to \infty$, by using the Markov's inequality, the Rosenthal's inequality and the above lemma,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{\theta\in[-1,1]} |M_{n1}(\theta) - M_{n1}^{(k_0)}(\theta)| \geq \varepsilon\right] \\
&\leq \frac{C}{\varepsilon n}\mathbb{E}\left[\max_{-2^{k_0}\leq j\leq 2^{k_0}} W_n(j,k_0)\right] \\
&\leq \frac{C2^{k_0/2}}{\varepsilon n}\max_{-2^{k_0}\leq j\leq 2^{k_0}}\left(\mathbb{E}\left[(W_n(j,k_0))^2\right]\right)^{1/2} \\
&\leq \frac{C2^{k_0/2}}{\varepsilon n}(1+n2^{-k_0}) \leq C_1\varepsilon^{-1}2^{-k_0/2},
\end{aligned}$$

which implies $(*)$ by taking k_0 large enough such that $\varepsilon^{-1}2^{-k_0/2} \leq \varepsilon_1$.

Application to Asymptotics of S_n under H_0

Corollary

Suppose that Assumptions 2.1-2.2 hold. For any locally bounded functions $g_1(x)$, $g_2(x)$ and $g_3(x)$ on $\mathbb{R}$, we have

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} u_k \mathbb{I}(x_k/d_n), \frac{1}{n}\sum_{k=1}^{n} g_1(x_k/d_n)\mathbb{I}(x_k/d_n), \frac{1}{\sqrt{n}}\sum_{k=1}^{n} u_k g_2(x_k/d_n), \frac{1}{n}\sum_{k=1}^{n} g_3(x_k/d_n)\right)$$

$$\Rightarrow \left(\int_0^1 \mathbb{I}(X_t \le x)dB_{3t}, \int_0^1 g_1(X_t)\mathbb{I}(X_t \le x)dt, \int_0^1 g_2(X_t)dB_{3t}, \int_0^1 g_3(X_t)dt\right),$$

on $D_{\mathbb{R}^4}[-\infty, \infty]$.

Proof. By choosing $y_{nk} = x_k/d_n$.

Proof of Theorem 2.1

Theorem 2.1 Suppose that Assumptions 2.1-2.3 hold. Then under H_0

$$S_n \to_D \sup_{x\in\mathbb{R}} |\alpha(x)|,$$

where

$$S_n = \sup_{x\in\mathbb{R}} |\alpha_n(x)| = \sup_{x\in\mathbb{R}} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left[y_k - g(x_k, \widehat{\theta}_n) \right] \mathbb{1}(x_k \le x) \right|,$$

and where $\alpha(x) = \int_0^1 \mathbb{1}(X_t \le x) dB_{3t}$.

Proof of Theorem 2.1 (cont.)

$$\begin{aligned}\alpha_n(x) =& \frac{1}{\sqrt{n}} \sum_{k=1}^{n} u_k \mathbb{1}(x_k/d_n \le x) \\ & - \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left[g(x_k, \widehat{\theta}_n) - g(x_k, \theta_0 \right] \mathbb{1}(x_k/d_n \le x) \\ :=& \alpha_{n1}(x) - \alpha_{n2}(x).\end{aligned}$$

We can prove

- $\alpha_{n1}(x) \Rightarrow \alpha(x)$ on $D_{\mathbb{R}}[-\infty, \infty]$, by the weak convergence theorem for martingales/the corollary.
- $\sup_{x \in \mathbb{R}} |\alpha_{n2}(x)| = o_p(1)$.
- Theorem 2.1 follows from the above two facts and the continuous mapping theorem.

Some Special Examples

Let $\sigma^2 > 0$ be the asymptotic variance of $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} u_k$. We may write the covariance matrix in Assumption 2.2 explicitly as

$$\Omega = \begin{pmatrix} 1 & 0 & \rho_1\sigma \\ 0 & 1 & \rho_2\sigma \\ \rho_1\sigma & \rho_2\sigma & \sigma^2 \end{pmatrix},$$

recall

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \varepsilon_k, \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \varepsilon_{-k}, \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} u_k \right) \Rightarrow (B_{1t}, B_{2t}, B_{3t}), \text{ on } D_{\mathbb{R}^3}[0, \infty).$$

Clearly, the limiting null distribution of S_n heavily depends on the value of ρ_1 and ρ_2, i.e., the dependent structure between $\{X_t\}$ and $\{B_{3t}\}$.

Some Special Examples: Case 1

Case 1: $\rho_1 = \rho_2 = 0$. We may choose $u_t = \sigma_t \eta_t$, where $\{\eta_j\}_{j\in\mathbb{Z}}$ is independent of $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and is a sequence of i.i.d. random variables with $\mathbb{E}\eta_1 = 0$, $\mathbb{E}\eta_1^2 = 1$, and $\mathbb{E}|\eta_1|^{2+\gamma} < \infty$ for some $\gamma > 0$, and where

$$\sigma_t \in \sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots; \eta_{t-1}, \eta_{t-2}, \ldots)$$

is a stationary process with $\sigma^2 = \mathbb{E}\sigma_t^2 < \infty$.

Some Special Examples: Case 1 (Wang, W., Zhu, 2018)

Corollary

Let u_t be defined as above. Suppose that Assumptions 2.1 and 2.3 hold. Then, under H_0,

$$S_n \to_D \sigma \sup_{t \in [0,1]} |B_t|,$$

where $\{B_t\}$ is a standard Brownian motion.

Some Special Examples: Case 2

Case 2: $\rho_1 = 1,\ \rho_2 = 0$. We may choose $u_t = \varepsilon_{t+1}\sigma_t$ where

$$\sigma_t \in \sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$$

is a stationary process with $\sigma^2 = \mathbb{E}\sigma_t^2 < \infty$.

Some Special Examples: Case 2 (Wang, W., Zhu, 2018)

Corollary

Let u_t be defined as above. Suppose that Assumptions 2.1 holds with $\tau = 0$ and ξ_t satisfying **SM** *condition, and Assumption 2.3 holds. Then, under H_0,*

$$S_n \to_D \sigma \sup_{x \in \mathbb{R}} \left| \int_0^1 \mathbb{1}(B_t \leq x) dB_t \right|,$$

where $\{B_t\}$ is a standard Brownian motion.

Conclusion Remark

- The simulation results agree well with our derivation.
- We also study a real problem: CO2(Carbon Dioxide) Emission v.s. GDP among several countries. Our method also performs well.

Thank You!